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1557

CONTENTS.

Page

On the Instability of the Pear-shaped Figure of Equilibrium of a rotating mass of Homogeneous Liquid—(*contd.*): S. R. U. Savoor ... ... 25—48

NOTES AND QUESTIONS:

A Test for the Convergency and Divergency of a Series of Positive Terms: T. Vijayaraghavan *17*

An Elementary Definite Integral: G. S. Mahajani ... *22*

Solutions *27*

Questions for Solution *30*

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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§ 7. *Explicit expressions for U_n.*

Since $$U = \text{Lim} \int\!\!\int K(u + \zeta,\ v + \zeta')\, dv\, d\psi',$$
it is easy to see that its expansion is

$$U = \text{Lim} \int\!\!\int \Big[K(uv) + \Big(\zeta \frac{\partial}{\partial u} + \zeta' \frac{\partial}{\partial v} \Big) K(u\, v) + \dots\dots\dots$$

$$+ \frac{1}{n!} \Big(\zeta \frac{\partial}{\partial u} + \zeta' \frac{\partial}{\partial v} \Big)^n K(u\, v) + \dots\dots \Big] dv\, d\psi'$$

$$= U_0 + U_1 + \dots\dots\dots + U_n + \dots\dots\dots \text{ say.}$$

Hence we have the expression

$$U_n = \text{Lim} \int\!\!\int \frac{1}{n!} \Big(\zeta \frac{\partial}{\partial u} + \zeta' \frac{\partial}{\partial v} \Big)^n K(u\, v)\, dv\, d\psi'$$

$$= \text{Lim} \int\!\!\int \Sigma \frac{\zeta^i \zeta'^j}{i!\, j!} \frac{\partial^n}{\partial u^i\, \partial v^j} K(u\, v)\, dv\, d\psi', \tag{7.1}$$

wherein $i + j = n$; $i = 0, 1, \dots\dots\dots n$
$j = 0, 1 \dots\dots\dots n.$

In this, consider the term for which $j = 0$. It is

$$\frac{\zeta^n}{n!} \text{Lim} \frac{\partial^n}{\partial u^n} \int\!\!\int K(u\, v)\, dv\, d\psi'$$

since ζ is a function of ψ and is therefore independent of the integration with respect to ψ'.

But $\int\!\!\int K(u\, v)\, dv\, d\psi'$ = potential of the elliptic cylinder at the point u.*

Since the point u is inside the cylinder, we have

$$\text{R. H. S.} = \text{constant} - \frac{2\pi\,(bx^2 + ay^2)}{a + b},$$

where $$x = \sqrt{(u + q)} \cos\psi\,;\ y = \sqrt{u} \sin\psi.$$

Since $$U_0 = \underset{u \to \rho - 0}{\text{Lim}} U$$

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* In general we shall say 'the point u' when we mean the point whose co-ordinates are

$$\sqrt{(u + q)} \cos\psi,\ \sqrt{u} \sin\psi.$$

4

195323

at an internal point, we get

$$\left.\begin{aligned} &\lim_{\substack{v=\rho \\ u\to\rho-0}} \frac{\partial^n}{\partial u^n} \iint K(uv)\, dv\, d\psi' = 0, \text{ for } n > 1, \text{ and} \\ &\qquad = -\frac{2\pi(b\cos^2\psi + a\sin^2\psi)}{a+b}, \\ &\qquad\qquad \text{for } n = 1. \end{aligned}\right\} \quad (7.2)$$

Thus all the terms in U which depend only on ζ except that coming in U_1, are zero and we have for U_n the expression

$$U_n = \lim_{\substack{v=\rho \\ u\to\rho-0}} \sum \frac{\zeta^i}{i!\,j!} \frac{\partial^n}{\partial u^i\, \partial v^j} \iint K(uv)\, \zeta'^j\, dv\, d\psi'$$

wherein $i + j = n$, $j \neq 0$,

i.e., $$U_n = \lim \sum \frac{\zeta^i}{i!\,j!} \frac{\partial^{n-1}}{\partial u^i\, \partial v^{j-1}} \int K(uv)\, \zeta'^j\, d\psi'$$

where $i + j = n$ and $j \neq 0$;

or $$U_n = \lim_{\substack{v=\rho \\ v\to\rho-0}} \left\{ \begin{aligned} &\frac{1}{n!} \frac{\partial^{n-1}}{\partial v^{n-1}} \int K\zeta'^n\, d\psi' + \ldots\ldots\ldots\ldots \\ &+ \frac{\zeta^r}{r!\,(n-r)!} \frac{\partial^{n-1}}{\partial u^r\, \partial v^{n-r-1}} \int K\zeta'^{n-r} d\psi' + . \\ &\ldots\ldots + \frac{\zeta^{n-1}}{1!\,(n-1)!} \frac{\partial^{n-1}}{\partial u^{n-1}} \int K\zeta'\, d\psi \end{aligned} \right\} \quad (7.3)$$

wherein we have put K for $K(uv)$ for brevity.

§ 8. In the above, we have considered the case $n > 1$. Consider now the form for $n = 1$. From (7.1) we have

$$U_1 = \lim \iint \left(\zeta \frac{\partial}{\partial u} + \zeta' \frac{\partial}{\partial v}\right) K(uv)\, dv\, d\psi'$$

$$= \zeta \lim \frac{\partial}{\partial u} \iint K(uv)\, dv\, d\psi' + \lim \int K(uv)\, \zeta'\, d\psi'.$$

Using equation (7.2) we get

$$U_1 = \lim \int K(uv)\, \zeta'\, d\psi' - \zeta\, \frac{2\pi(b\cos^2\psi + a\sin^2\psi)}{a+b}$$

$$\equiv \int K(\rho\rho)\, \zeta'\, d\psi' - \zeta \left\{ \frac{2\pi ab}{(a+b)^2} + \frac{2\pi(a^2\sin^2\psi + b^2\cos^2\psi)}{(a+b)^2} \right\} \quad (8.1)$$

§ 9. Let us now go back to equations (5.3), (5.4), (5.5). By our definition of the new figure we have equation (5.5), *i.e.*,

$$U + \frac{\omega^2}{2}(x^2 + y^2) \equiv K\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \text{constant}$$

on its surface which is given by

$$x = \sqrt{(\rho + q + \zeta)} \cos \psi, \; y = \sqrt{(\rho + \zeta)} \sin \psi.$$

Hence remembering that U can be expanded, we get

$$U_0 + U_1 + \ldots\ldots + U_n + \ldots\ldots + \frac{\omega^2}{2}(\rho + q \cos^2 \psi + \zeta)$$

$$\equiv K\left(1 + \frac{H\zeta}{\Delta^2}\right) + \text{constant},$$

where $H \equiv H(\rho\psi) = \rho + q \sin^2 \psi\, ; \; \Delta \equiv \Delta(\rho) = \sqrt{\rho(\rho + q)}$; (9.1)
or, on putting in the value of U_1 from (8.1) and that of ω^2 from (4.3), we get

$$U_0 + \frac{\omega^2}{2}(\rho + q \cos^2 \psi) + \int K(\rho\rho)\, \zeta'\, d\psi' - \frac{2\pi H\zeta}{(a + b)^2} + U_2 + U_3 + \ldots$$

$$= K\left(1 + \frac{H\zeta}{\Delta^2}\right) + \text{constant}\, ;$$

or, since $\quad U_0 + \dfrac{\omega^2}{2}(\rho + q \cos^2 \psi) = \text{constant}$

on the ellipse, we get the fundamental equation

$$\frac{2ab}{(a + b)^2} H\zeta + \frac{1}{2\pi} \int \log R^2(\rho\rho)\, H'\zeta'\, d\psi'$$

$$= \frac{ab}{\pi}(U_2 + U_3 + \ldots\ldots) - LH\zeta + \text{constant}, \quad (9.2)$$

where $\quad L = \dfrac{K}{\pi\Delta}$. (9.3)

Let us now express the other fundamental relation, *viz.* that the volume of the liquid is constant.

Since the element of volume of the new figure is

$$d\,(\text{volume}) = \tfrac{1}{2}\, G(v + \zeta', \; \psi')\, dv\, d\psi'$$

$$\equiv \tfrac{1}{2}\, \frac{H(v + \zeta', \; \psi')}{\Delta(v + \zeta')}\, dv\, d\psi',$$

the condition that the volume remains the same is

$$\lim_{v=\rho} \tfrac{1}{2} \int\int \frac{H(v+\zeta', \psi')}{\Delta(v+\zeta')} \, dv \, d\psi' = \text{constant} = \text{volume of elliptic cylinder};$$

expanding as before

$$\lim_{v=\rho} \int\int \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} \frac{\partial^n}{\partial v} \left\{ \frac{H(v\psi)}{\Delta(v)} \right\} dv \, d\psi = 0;$$

or

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int \frac{\partial^n}{\partial \rho^n} \left[\frac{H}{\Delta} \right] \zeta^{n+1} \, d\psi = 0. \tag{9.4}$$

§10. The equations (9.2) and (9.4) are the two equations that we must use to find ζ as a function of ψ. Let us try to do this by finding for ζ a power series of which the co-efficients can be calculated successively. For this purpose let us suppose that ζ encloses an arbitrary parameter α such that $|\alpha|$ being small we can expand ζ in a power series.

$$\zeta = \zeta_1 \alpha + \zeta_2 \alpha^2 + \ldots\ldots + \zeta_n \alpha^n + \ldots\ldots \tag{10.1}$$

where $\zeta_1 \neq 0$ and the ζ_n's are functions of ψ.

Let us also assume for the present that we can find positive numbers l_n such that for all values of ψ we have $|\zeta_n| < l_n$ and that the power series

$$l_1 \alpha + l_2 \alpha^2 + l_3 \alpha^3 + \ldots\ldots$$

is convergent for sufficiently small values of α.

Then the assumed expansion for ζ will be absolutely convergent and uniformly convergent with respect to ψ. Let us suppose also that the undetermined constant L is also a function of α given by

$$L = L_0 + L_1 \alpha + L_2 \alpha^2 + \ldots\ldots$$

where the L_n's are independent of α.

Since the series for ζ is absolutely and uniformly convergent at least so long as the series

$$l_1 \alpha + l_2 \alpha^2 + \ldots\ldots + l_n \alpha^n + \ldots\ldots$$

is convergent we may substitute it in any convergent series whose terms are functions of ζ and re-arrange it in powers of α. The re-arranged series will still be convergent.

Let us do so for the equation (9.4). Then we shall get a power series in α identically equal to zero.

Hence its various co-efficients must be zero.

Let us for the moment confine ourselves to the co-efficient of α

This gives at once

$$\int H \zeta_1 \, d\psi = 0. \tag{10.2}$$

since $\Delta \equiv \Delta(\rho)$ and is independent of ψ.

Similarly if we do the same thing for the equation (9.2) we get

$$\frac{2ab}{(a+b)^2} H \zeta_1 + \frac{1}{2\pi} \int \log R^2 (\rho\rho) \, H'\zeta'_1 \, d\psi' = -L_0 H \zeta_1. \tag{10.3}$$

since U_2, U_3 ...... are all of higher order than the first.

Hence multiplying both sides by $\begin{pmatrix}\cos\\ \sin\end{pmatrix} n\psi \, d\psi$ and integrating and using the formulæ (2.6), (2.7), we get

$$\left.\begin{aligned} \left[\frac{2ab}{(a+b)^2} - \frac{1+e^{-2n\alpha}}{n} + L_0\right] \int H\zeta_1 \cos n\psi \, d\psi &= 0, \\ \left[\frac{2ab}{(a+b)^2} - \frac{1-e^{-2n\alpha}}{n} + L_0\right] \int H\zeta_1 \sin n\psi \, d\psi &= 0; \end{aligned}\right\} \tag{10.4}$$

or if we put

$$t \equiv \frac{a-b}{a+b} = e^{-2\alpha}; \quad \left\{\begin{aligned} T_n &= \tfrac{1}{2}(1-t^2) - \frac{1}{n}(1+t^n) \\ T_n' &= \tfrac{1}{2}(1-t^2) - \frac{1}{n}(1-t^n) \end{aligned}\right\} \tag{10.5}$$

$$(T_n + L_0) \int H\zeta_1 \cos n\psi \, d\psi = 0$$
$$= (T'_n + L_0) \int H\zeta_1 \sin n\psi \, d\psi.$$

Hence

$$\left.\begin{aligned} &\int H\zeta_1 \cos n\psi \, d\psi \\ &\int H\zeta_1 \sin n\psi \, d\psi \end{aligned}\right\} \equiv 0$$

all the time that $T_n + L_0 \neq 0$, $T'_n + L_0 \neq 0$.

In the above we have considered $n \geqslant 1$.

But equation (10.2) gives

$$\int H \zeta_1 \, d\psi = 0.$$

Hence all the time $T_n + L_0 \neq 0$, $T'_n + L_0 \neq 0$, we must have

$$\int H \zeta_1 \cos n\psi \, d\psi \equiv 0 \equiv \int H \zeta_1 \sin n\psi \, d\psi.$$

Thus we see that if all the $T_n + L_0$, $T'_n + L_0$ were different from zero, we shall have $\zeta_1 \equiv 0$.

But by our hopothesis (10.1), ζ_1 is not equal to zero identically. Hence we conclude that at least one of the

$$T_n + L_0, \; T'_n + L_0$$

must be equal to zero identically.

Suppose it is $T_m + L_0$. (10.6)

Then all the other $T_n + L_0$, $T'_n + L_0$ are different from zero.

§ **11.** So long we have not defined what α is. We merely said that it was an arbitrary parameter on which ζ depends. Let us now define it for the sake of precision by the equation,

$$\alpha = \frac{1}{\pi} \int H \zeta \cos m\psi \, d\psi. \qquad (11.1)$$

Substituting in the expansion (10.1) for ζ and equating powers of α, we get

$$1 = \frac{1}{\pi} \int H \zeta_1 \cos m\psi \, d\psi \qquad (11.2)$$

and for $n \neq 1$,

$$0 = \int H \zeta_n \cos m\psi \, d\psi. \qquad (11.3)$$

From this it is obvious that

$$H \zeta_1 = \cos m\psi \qquad (11.4)$$

and for $n > 1$

$$\int H \zeta_n \cos m\psi \, d\psi = 0;$$

we shall require these results later on.

§ 12. Let us now return to our fundamental equation (9.2), *viz.*

$$\frac{2ab}{(a+b)^2} \mathrm{H}\zeta + \frac{1}{2\pi} \int \log \mathrm{R}^2(\rho\rho)\, \mathrm{H}'\zeta'\, d\psi'$$

$$= \frac{\Delta}{\pi}(\mathrm{U}_2 + \mathrm{U}_3 + \ldots\ldots) - \mathrm{LH}\zeta + \text{constant}$$

and equate co-efficients of α^2 on both sides.

The left-hand side gives

$$\frac{2ab}{(a+b)^2} \mathrm{H}\zeta_2 + \frac{1}{2\pi} \int \log \mathrm{R}^2(\rho\rho)\, \mathrm{H}'\zeta'_2\, d\psi'.$$

Since U_n is of order n in ζ, ζ' on the right-hand side, we need consider only the part

$$\frac{\Delta}{\pi}\mathrm{U}_2 - \mathrm{LH}\zeta$$

to get the co-efficient of α^2.

The full expression for this is

$$\frac{\Delta}{\pi} \operatorname{Lim} \left[\tfrac{1}{2} \frac{\partial}{\partial v} \int \mathrm{K}(uv)\, \zeta'^2\, d\psi' + \zeta \frac{\partial}{\partial u} \int \mathrm{K}(uv)\, \zeta'\, d\psi'\right]$$

$$-(\mathrm{L}_0 + \mathrm{L}_1\, \alpha + \mathrm{L}_2\, \alpha^2 + \ldots\ldots)\, \mathrm{H}(\zeta_1\, \alpha + \zeta_2\, \alpha^2 + \ldots\ldots).$$

The co-efficient of α^2 in this is

$$\frac{\Delta}{\pi}(\zeta_1, \zeta_1) - \mathrm{L}_0\, \mathrm{H}\zeta_2 - \mathrm{L}_1\, \mathrm{H}\zeta_1,$$

where $$(\zeta_1, \zeta_1) = \operatorname{Lim} \left[\tfrac{1}{2} \frac{\partial}{\partial v} \int \mathrm{K}(uv)\, \zeta_1'^2\, d\psi' + \zeta_1 \frac{\partial}{\partial u} \int \mathrm{K}(uv)\, \zeta_1'\, d\psi'\right]; \qquad (12.1)$$

equating the co-efficient of α^2 on both sides, we get the equation

$$\frac{2ab}{(a+b)^2} \mathrm{H}\zeta_2 + \frac{1}{2\pi} \int \log \mathrm{R}^2(\rho\rho)\, \mathrm{H}'\zeta_2'\, d\psi'$$

$$= \frac{\Delta}{\pi}(\zeta_1, \zeta_1) - \mathrm{L}_0\, \mathrm{H}\zeta_2 - \mathrm{L}_1\, \mathrm{H}\zeta_1; \quad (12.2)$$

or transposing the term $\mathrm{L}_0\, \mathrm{H}\zeta_2$ to the left-hand side, we get

$$\left[\frac{2ab}{(a+b)^2} - \mathrm{T}_m\right] \mathrm{H}\zeta_2 + \frac{1}{2\pi} \int \log \mathrm{R}^2(\rho\rho)\, \mathrm{H}'\zeta'_2\, d\psi'$$

$$= \frac{\Delta}{\pi}(\zeta_1, \zeta_1) - \mathrm{L}_1\, \mathrm{H}\zeta_1 \qquad (12.3)$$

since $\mathrm{L}_0 = -\mathrm{T}_m$ by (10.6).

Multiply by $\cos l\psi d\psi$ or $\sin l\psi d\psi$ and integrate using the formulæ (2.6), (2.7) and we get

$$(T_l - T_m) \int H\zeta_2 \cos l\psi \, d\psi$$

$$= \frac{\Delta}{\pi} \int (\zeta_1, \zeta_1) \cos l\psi \, d\psi - L_1 \int H\zeta_1 \cos l\psi \, d\psi; \quad (12.4)$$

or $\quad (T'_l - T_m) \int H\zeta_2 \sin l\psi \, d\psi$

$$= \frac{\Delta}{\pi} \int (\zeta_1, \zeta_1) \sin l\psi \, d\psi - L_1 \int H\zeta_1 \sin l\psi \, d\psi.$$

Now $\qquad H\zeta_1 = \cos m\psi$ by (11.4).

Hence, if $l \neq m$, we get

$$\left.\begin{aligned} (T_l - T_m) \int H\zeta_2 \cos l\psi \, d\psi &= \frac{\Delta}{\pi} \int (\zeta_1, \zeta_1) \cos l\psi \, d\psi, \\ (T_l' - T_m) \int H\zeta_2 \sin l\psi \, d\psi &= \frac{\Delta}{\pi} \int (\zeta_1, \zeta_1) \sin l\psi \, d\psi; \end{aligned}\right\} \quad (12.5)$$

and when $l = m$, we get

$$\left.\begin{aligned} L_1 \int \cos^2 m\psi \, d\psi &= \frac{\Delta}{\pi} \int (\zeta_1, \zeta_1) \cos m\psi \, d\psi; \\ (T'_m - T_m) \int H\zeta_2 \sin m\psi \, d\psi &= \frac{\Delta}{\pi} \int (\zeta_1, \zeta_1) \sin m\psi \, d\psi. \end{aligned}\right\} \quad (12.6)$$

Thus we get the values of the integrals

$$\int H\zeta_2 \cos l\psi \, d\psi, \int H\zeta_2 \sin l\psi \, d\psi$$

for all values of l (except $l = 0$) and that of the first for $l = m$, since

$$T_l \neq T_m, \; T_l' \neq T_m$$

in all other cases.

But when $l = 0$ the integral

$$\int H\zeta_2 \, d\psi$$

is given by the equation (9.4); and when $l = m$ we have by (11.3)

$$\int H\zeta_2 \cos m\psi \, d\psi = 0.$$

Hence we know the values of the integrals

$$\int H\zeta_2 \cos l\psi \, d\psi, \int H\zeta_2 \sin l\psi \, d\psi$$

for all values of l and so the function $H\zeta_2$ can be found as a Fourier Series.

So can ζ_2 be found without any ambiguity.

§ 13. In exactly the same way, we can calculate successively all the ζ_n and the L_{n-1}, the ζ_n, L_{n-1} being calculated at the same time. This process can be carried as far as we please. For, in order to find the co-efficient of α^n in

$$\frac{\Delta}{\pi}(U_2 + U_3 + \ldots\ldots + U_n + \ldots\ldots) - LH\zeta$$

we need only take

$$\zeta_1 \alpha + \zeta_2 \alpha^2 + \ldots\ldots + \zeta_{n-1} \alpha^{n-1}$$

for ζ in the expressions for

$$U_2, U_3, \ldots\ldots U_n \ldots\ldots$$

and therefore the co-efficient of α^n on the right-hand side of equation (9.2) will involve only $\zeta_1, \zeta_2 \ldots\ldots \zeta_{n-1}$ which we assume to have been found and ζ_n in one single term, *viz.* $-L_0 H\zeta_n$ which can be transposed. The left-hand side of the same equation contains only ζ_n. So when $\zeta_1 \ldots\ldots \zeta_{n-1}$ have all been calculated we can find ζ_n, L_{n-1} at the same time. Thus we have shown how the co-efficients, ζ_n of the series for ζ can be found uniquely and without ambiguity.

Let us note however that we have in the above taken the simplest possible functions ζ_n which satisfy our equations. They are, in fact, the particular integrals of our equations. For the most general solutions of the ζ_n contain certain definite functions which occur because of the presence of the arbitrary constants that appear in the most general solutions of the functions

$$\zeta_1, \zeta_2 \ldots\ldots \zeta_{n-1}.$$

Thus for example we have shown that the particular solution

$$\zeta_1 = \frac{\cos m\psi}{H} \equiv z$$

satisfies the equation for ζ_1, *viz.*

$$\frac{2ab}{(a+b)^2} H\zeta_1 + \frac{1}{2\pi}\int \log R^2(\rho\rho) H'\zeta_1' \, d\psi' = \text{constant.}$$

Hence the most general solution for ζ_1 is

$$\zeta_1 = z + \frac{c_1}{H}.$$

Hence the most general solution for ζ_2 will contain the particular integral τ_1, a definite function due to the presence of c_1 in ζ_1 and $\frac{c_2}{H}$.

The most general solution for ζ_n will therefore be of the form

$$\left.\begin{aligned}\zeta_n = \tau_{n-1} &+ (c_1\,\phi_{n,1} + c_1^{2}\,\phi_{n,2} + \ldots\ldots + c_1^{n-1}\,\phi_{n,n-1}) \\ &+ (c_2\,\chi_{n,2} + c_2^{2}\,\chi_{n,2} + \ldots\ldots + c_2^{n-2}\,\chi_{n,n-2}) \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ (c_r\,\theta_{n,1} + \ldots\ldots + c_r^{n-r}\,\theta_{n,n-r}) + \ldots\ldots + \frac{c_n}{H}\end{aligned}\right\} \quad (13.1)$$

wherein some of the functions ϕ, χ, θ, etc. for the case n are the same as some of the similar functions for the case $(n-r)$ and others are new functions and so on.

But when we put the constants c_r equal to zero, we get the particular integrals of our equations and these are the ones which have the property of being unique and unambiguous that was mentioned above.

Let us note also that the particular integrals so chosen have this property that in the Fourier expansion of $H\tau_{n-1}$ for $n > 1$ the term in $\cos m\psi$ is absent*.

In other words, τ_{n-1} is a particular solution of the equation

$$\frac{2ab}{(a+b)^2}\,H\zeta_n + \frac{1}{2\pi}\int \log R^2\,(\rho\rho)\,H'\zeta'_n\,d\psi' \qquad (13.2)$$

$$= \text{co-efficient of } \alpha^n \text{ in } \frac{\Delta}{\pi}\,(U_2 + U_3 + \ldots\ldots + U_n + \ldots\ldots)$$
$$- LH\zeta + C\cos m\psi + \text{constant}$$
$$= \text{co-efficient of } \alpha^n \text{ in } \frac{\Delta}{\pi}\,W + C\cos m\psi + \text{constant, say}$$

where $-\frac{\pi}{\Delta}\,C$ is the co-efficient of $\cos m\psi$ in the Fourier expansion of the co-efficient of α^n in $\frac{\Delta}{\pi}\,W$.

This particular solution is taken in order to avoid infinities which will otherwise arise owing to the presence of the factor τ_m (which is zero) in the denominator of the term in $\cos m\psi$.

* Cf. the method used in § 17 where also this is done.

§ 14. Later on we shall have occasion to use the integral

$$\int (\zeta_1, \zeta_1) \cos l\psi \, d\psi. \tag{14 1}$$

Let us therefore evaluate it. It is evident that evaluating the above integral for the different values of l ranging from one to infinity is equivalent to expanding the function (ζ_1, ζ_1) in Fourier Series. Let us then find this Fourier Expansion.

We have from (12.1),

$$(\zeta_1, \zeta_1) = \lim_{\substack{v = \rho \\ u \to \rho - 0}} \left[\tfrac{1}{2} \frac{\partial}{\partial v} \int K(uv) \, \zeta_1'^2 \, d\psi' + \zeta_1 \frac{\partial}{\partial u} \int K(uv) \, \zeta_1' \, d\psi' \right]$$

wherein we have by (6.3), (11.4)

$$K(uv) = -\log R^2(uv) \frac{H(v\psi')}{2\Delta(v)}; \quad H\zeta_1 \equiv H(\rho\psi)\,\zeta_1 = \cos m\psi$$

Hence

$$(\zeta_1, \zeta_1) = \lim \left\{ \begin{aligned} &\tfrac{1}{2} \frac{\partial}{\partial v} \int -\frac{\log R^2(uv)}{2\Delta(v)} H(v\psi') \frac{\cos^2 m\psi'}{[H(\rho\psi')]^2} \, d\psi' \\ &+ \frac{\cos m\psi}{H(\rho\psi)} \frac{\partial}{\partial u} \int \frac{-\log R^2(uv)}{2\Delta(v)} H(v\psi') \frac{\cos m\psi'}{H(\rho\psi')} \, d\psi \end{aligned} \right\}$$

$$= I_1 + I_2 \text{ say.}$$

Consider I_2 first. Since the differentiation is with respect to u we may put $v = \rho$ in this for the limit.

Hence $$I_2 = \frac{\cos m\psi}{H(\rho\psi)} \lim_{u \to \rho - 0} \frac{\partial}{\partial u} \int \frac{-\log R^2(u\rho)}{2\Delta(\rho)} \cos m\psi' \, d\psi'.$$

But by (2.6)

$$\int \frac{\log R^2(uv)}{\Delta(v)} \cos k\psi' \, d\psi' = -\frac{2\pi}{k} \frac{e^{-k\alpha_v}}{\Delta(v)} \left(e^{k\alpha_u} + e^{-k\alpha_u} \right) \cos k\psi$$

for $u < v$.

Hence if we put

$$P_k(t) = e^{k\alpha_t} + e^{-k\alpha_t}$$

$$Q_k(t) = \frac{e^{-k\alpha_t}}{\Delta(t)} \tag{14.2}$$

so that for $u < v$ the equation (2.6) may be written

$$\int \frac{\log R^2(uv)}{\Delta(v)} \cos k\psi' \, d\psi' = -\frac{2\pi}{k} P_k(u) Q_k(v) \cos k\psi$$

we see that

$$I_2 = \frac{\cos m\psi}{H(\rho\psi)} \cdot \lim_{u \to \rho - 0} \frac{\partial}{\partial u} \cdot \frac{2\pi}{m} \cdot \tfrac{1}{2} P_m(u) Q_m(\rho) \cos m\psi. \quad (14.3)$$

For the sake of brevity let us drop the suffix m wherever it occurs and also the argument of the functions when it is ρ so that

$$I_2 = \frac{\pi}{m} \frac{dP}{d\rho} Q \cdot \frac{\cos^2 m\psi}{H(\rho\psi)}. \quad (14.4)$$

Consider now I_1. It is of the form

$$\lim_{\substack{v = \rho \\ u \to \rho - 0}} \tfrac{1}{2} \frac{\partial}{\partial v} \int X(uv)\, Y(v)\, Z(\rho)\, d\psi'.$$

Hence

$$I_1 = \tfrac{1}{2} \lim_{u \to \rho} \int X(u\rho) \left(\frac{\partial Y}{\partial v}\right)_{v = \rho} Z(\rho)\, d\psi'$$

$$+ \tfrac{1}{2} \lim_{\substack{v = \rho \\ u \to \rho - 0}} \frac{\partial}{\partial v} \int X(uv)\, Y(\rho)\, Z(\rho)\, d\psi'.$$

In this, put

$$Y(v) = H(v\psi') = v + q \sin^2 \psi'; \quad X(uv) = -\frac{\log R^2(uv)}{2\Delta(v)}.$$

Hence we get

$$I_1 = \tfrac{1}{4} \lim_{u \to \rho} \int -\frac{\log R^2(u\rho)}{\Delta(\rho)} \frac{\cos^2 m\psi'}{[H(\rho\psi')]^2} d\psi'$$

$$+ \tfrac{1}{4} \lim_{\substack{v = \rho \\ u \to \rho - 0}} \frac{\partial}{\partial v} \int -\frac{\log R^2(uv)}{\Delta(v)} \frac{\cos^2 m\psi'}{H(\rho\psi')} d\psi'.$$

If now we put

$$\tau \cos m\psi \equiv \frac{\cos^2 m\psi}{H(\rho\psi)} \equiv \frac{\cos^2 m\psi}{\rho + q \sin^2 \psi} = \sum_{i=0}^{\infty} a_i \cos 2i\psi \quad (14.5)$$

(noting that the terms with cosines of odd multiples of ψ cannot be present) so that

$$\frac{\cos^2 m\psi}{[H(\rho\psi)]^3} = -\sum_{i=0}^{\infty} \frac{da_i}{d\rho} \cos 2i\psi \tag{14.6}$$

we get

$$I_1 = \tfrac{1}{4} \lim_{u \to \rho} \int \frac{\log R^2(u\rho)}{\Delta(\rho)} \sum \frac{da_i}{d\rho} \cos 2i\psi' \, d\psi'$$

$$+ \tfrac{1}{4} \lim_{\substack{v=\rho \\ u \to \rho - 0}} \frac{\partial}{\partial v} \int - \frac{\log R^2(uv)}{\Delta(v)} \sum a_i \cos 2i\psi' \, d\psi'$$

$$= \text{constant} + \tfrac{1}{4} \sum_{i=1}^{\infty} -\frac{2\pi}{2i} P_{2i}(\rho) Q_{2i}(\rho) \cos 2i\psi$$

$$+ \tfrac{1}{4} \lim \frac{\partial}{\partial v} \sum_{i=1}^{\infty} \frac{2\pi}{2i} P_{2i}(u) Q_{2i}(v) \cos 2i\psi$$

$$= \frac{\pi}{4} \sum_{i=1}^{\infty} \frac{P_{2i}}{i} \left[a_i \frac{dQ_{2i}}{d\rho} - Q_{2i} \frac{da_i}{d\rho} \right] \cos 2i\psi \tag{14.7}$$

in which the infinite constant corresponding to $i = 0$ is left out of account [as it can always be absorbed in the constant on the right-hand side of the equation for ζ_n, here ζ_2. See (9.2), (13.2)].

Thus

$$(\zeta_1, \zeta_1) \equiv (\tau, \tau) = \pi \sum \left\{ \begin{array}{l} \frac{P_{2i}}{4i} \left(a_i \frac{dQ_{2i}}{d\rho} - Q_{2i} \frac{da_i}{d\rho} \right) \\ \qquad + \frac{1}{m} \frac{dP}{d\rho} Q a_i \end{array} \right\} \cos 2i\psi$$

$$= \pi \sum \bar{a}_i \cos 2i\psi \quad \text{say} \tag{14.8}$$

where $$\bar{a}_i = \frac{P_{2i}}{4i} \left(a_i \frac{dQ_{2i}}{d\rho} - Q_{2i} \frac{da_i}{d\rho} \right) + \frac{1}{m} \frac{dP}{d\rho} Q a_i, \tag{14.9}$$

But we have

$$T_m \equiv \tfrac{1}{2}(1-t^2) - \frac{1}{m}(1+t^m)$$

i.e.,

$$\frac{T_m}{\Delta} = \frac{\frac{1}{2}(1-t^2)}{\Delta} - \frac{1}{m}\frac{e^{-m\alpha}}{\Delta}(e^{m\alpha}+e^{-m\alpha})$$

$$= \tfrac{1}{2}\frac{(1-t^2)}{\Delta} - \frac{1}{m}\,QP. \tag{14.10}$$

Similarly

$$\frac{T_{2i}}{\Delta} = \tfrac{1}{2}\frac{(1-t^2)}{\Delta} - \frac{1}{2i}\,Q_{2i}\,P_{2i}.$$

Using (14.10) we have

$$\frac{Q}{m} = \frac{1}{P}\left[\frac{T_{2i}-T}{\Delta} + \frac{1}{2i}P_{2i}\,Q_{2i}\right] \tag{14.11}$$

wherein the suffix m from T_m has been omitted for brevity.

Hence

$$\bar{a}_i = \frac{P_{2i}}{4i}\left(a_i\frac{dQ_{2i}}{d\rho} - Q_{2i}\frac{da_i}{d\rho}\right) + a_i\frac{dP}{d\rho}\frac{1}{P}\left[\frac{T_{2i}-T}{\Delta}+\frac{1}{2i}P_{2i}\,Q_{2i}\right]$$

$$= \frac{P_{2i}}{4iP^2}\left[a_i\frac{d(P^2Q_{2i})}{d\rho} - P^2Q_{2i}\frac{da_i}{d\rho}\right] + \frac{a_i}{P}\frac{dP}{d\rho}\frac{T_{2i}-T}{\Delta};$$

or if we put

$$a'_i = a_i\frac{d(P^2Q_{2i})}{d\rho} - P^2Q_{2i}\frac{da_i}{d\rho}, \tag{14.12}$$

we can write

$$(\tau,\tau) \equiv (\zeta_1,\zeta_1) = \pi\sum_i \bar{a}_i\cos 2i\psi$$

$$= \pi\sum_i\left[\frac{P_{2i}}{P^2}\frac{a'_i}{4i} + \frac{1}{P}\frac{dP}{d\rho}\cdot\frac{T_{2i}-T}{\Delta}a_i\right]\cos 2i\psi. \tag{14.13}$$

From this it is at once obvious that the integral (14.1), *viz.*

$$\frac{1}{\pi^2}\int(\zeta_1,\zeta_1)\cos l\psi\,d\psi$$

is zero when l is odd and that when $l = 2i$ its value is

$$\frac{P_{2i}}{P^2}\frac{a'_i}{4i} + \frac{1}{P}\frac{dP}{d\rho}\cdot\frac{T_{2i}-T}{\Delta}a_i. \tag{14.14}$$

§ 15. As we shall want the values of the constants a_i, a_i' later on, let us calculate them.

From (14.5) we have

$$\pi a_i = \int_0^{2\pi} \frac{\cos^2 m\psi \cos 2i\psi}{\rho + q \sin^2 \psi} d\psi \equiv \int_0^{2\pi} \frac{\cos^2 m\psi \cos 2i\psi}{a^2 \sin^2 \psi + b^2 \cos^2 \psi} d\psi.$$

Thus it is obvious that these integrals depend on

$$\int_0^{2\pi} \frac{\cos n\psi}{H(\rho\psi)} d\psi \equiv \int_0^{2\pi} \frac{\cos n\psi \, d\psi}{a^2 \sin^2 \psi + b^2 \cos^2 \psi}.$$

Let $$t = \frac{a-b}{a+b} \text{ so that } 0 < t < 1. \tag{15.1}$$

Then $$a^2 \sin^2 \psi + b^2 \cos^2 \psi = \frac{(a+b)^2}{4} (1 - 2t \cos 2\psi + t^2).$$

Hence using the well-known expansion

$$\frac{1-t^2}{1-2t\cos\theta + t^2} = 1 + 2t\cos\theta + 2t^2 \cos 2\theta + \ldots\ldots + 2t^n \cos n\theta + \ldots\ldots \tag{15.2}$$

we get

$$\frac{1}{a^2 \sin^2\psi + b^2 \cos^2\psi} = \frac{1}{ab}\Big[1 + 2t\cos 2\psi + \ldots\ldots + 2t^p \cos 2p\psi + \ldots\ldots\Big] \tag{15.3}$$

Hence we have

$$\int_0^{2\pi} \frac{\cos n\psi}{H(\rho\psi)} d\psi \equiv \int_0^{2\pi} \frac{\cos n\psi \, d\psi}{a^2 \sin^2 \psi + b^2 \cos^2 \psi} = 0, \text{ when } n \text{ is odd;}$$

and $$= \frac{2\pi}{ab}\left(\frac{a-b}{a+b}\right)^p = \frac{2\pi}{ab} t^p = \frac{2\pi t^p}{\Delta}, \tag{15.4}$$

if $n = 2p$.

Thus we get

$$a_i = \frac{1}{ab}\left[t^i + \tfrac{1}{2} t^{m+i} + \tfrac{1}{2} t^{|m-i|}\right]. \tag{15.5}$$

On differentiating with respect to ρ, we get

$$-\frac{da^i}{d\rho} \equiv -\left(\frac{1}{2a}\frac{\partial}{\partial a}+\frac{1}{2b}\frac{\partial}{\partial b}\right)\left[\frac{1}{ab}\left\{\left(\frac{a-b}{a+b}\right)^i+\ldots\ldots+\right\}\right] \quad (15.6)$$

$$=\frac{1}{2a^3b^3}\left\{\begin{array}{l} t_i(a^2+2iab+b^2)+\frac{1}{2}t^{m+i}(a^2+2\,\overline{m+i}\,ab+b^2) \\ \qquad +\frac{1}{2}t^{|m-i|}(a^2+2\,|m-i|\,ab+b^2)\end{array}\right\}$$

Let us now evaluate a'_i By (14.12)

$$a'_i = a_i\frac{d(\mathrm{P}^2\mathrm{Q}_{2i})}{d\rho} - \mathrm{P}^2\mathrm{Q}_{2i}\frac{da_i}{d\rho}.$$

Now $$\mathrm{P}=e^{m\alpha}+e^{-m\alpha};\ \mathrm{Q}_{2i}=\frac{e^{-2i\alpha}}{\Delta}.$$

Hence $$\frac{d(\mathrm{P}^2\mathrm{Q}_{2i})}{d\rho}=\frac{d}{d\rho}\left[\frac{e^{-2i\alpha}(e^{2m\alpha}+2+e^{-2m\alpha})}{\Delta}\right]$$

also $$\frac{d}{d\rho}\left[\frac{e^{2ka}}{\Delta}\right]=e^{2ka}\left[\frac{1}{\Delta}\cdot\frac{2k}{2\Delta}+\left(-\frac{1}{\Delta^2}\right)\left(\frac{2\rho+q}{2\Delta}\right)\right]$$

$$=\frac{e^{2k\alpha}}{2\Delta^2}\left[2k-\frac{a^2+b^2}{ab}\right]$$

Hence noting that $t^{-k}=e^{2k\alpha}$, we get

$$\frac{d(\mathrm{P}^2\mathrm{Q}_{2i})}{d\rho}=\frac{-(a^2+b^2)}{2a^3b^3}\left[t^{i-m}+2t^i+t^{m+i}\right]$$

$$+\frac{1}{a^2b^2}\left[(m-i)\,t^{i-m}-2i\,t^i-(m+i)\,t^{m+i}\right].$$

Thus using (15.5), (15.6), we get

$$a'_i=\frac{1}{ab}\left[t^i+\tfrac{1}{2}t^{m+i}+\tfrac{1}{2}t^{|m-i|}\right]\times$$

$$\left\{\begin{array}{l}\frac{1}{a^2b^2}\left\{(m-i)\,t^{i-m}-2i\,t^i-(m+i)\,t^{m+i}\right\} \\ \qquad -\frac{a^2+b^2}{2a^3b^3}\left\{t^{i-m}+2t^i+t^{m+i}\right\}\end{array}\right\}$$

$$+\frac{1}{ab}\left[t^{i-m}+2t^i+t^{m+i}\right]\frac{1}{2a^3b^3}\times$$

$$\left\{\begin{array}{l}t^i(a^2+2iab+b^2)+\frac{1}{2}t^{m+i}(a^2+2\,\overline{m+i}\,ab+b^2) \\ \qquad +\frac{1}{2}t^{|m-i|}(a^2+2\,|m-i|\,ab+b^2)\end{array}\right\}$$

In this, consider the co-efficient of

$$\frac{1}{2a^4b^4}\left[t^{i-m}+2t^i+t^{m+i}\right].$$

It is

$$\begin{aligned}&[t^i(a^2+2iab+b^2)+\tfrac{1}{2}t^{m+i}(a^2+2\overline{m+i}\,ab+b^2)\\ &\qquad+\tfrac{1}{2}t^{|m-i|}(a^2+2\,|m-i|\,ab+b^2)]\\ &\qquad-(a^2+b^2)\,[t^i+\tfrac{1}{2}t^{m+i}+\tfrac{1}{2}t^{|m-i|}]\\ &\equiv 2ab\,[it^i+\tfrac{1}{2}(m+i)\,t^{m+i}+\tfrac{1}{2}\,|m-i|\,t^{|m-i|}].\end{aligned}$$

The rest of a_i'

$$=\frac{1}{a^3b^3}\left[t^i+\tfrac{1}{2}t^{m+i}+\tfrac{1}{2}t^{|m-i|}\right]\left[(m-i)\,t^{i-m}-2i\,t^i-(m+i)\,t^{m+i}\right]$$

From this it is evident that when $i \geqslant m$ so that $|m-i| = i-m$, we have

$$a_i' \equiv 0; \tag{15.7}$$

also when $i < m$ so that $|m-i| = m-i$ we have

$$\begin{aligned}2a^3b^3\,a'_i = &\{2i\,t^i+(m+i)\,t^{m+i}\}\,\{t^{i-m}-t^{m-i}\}\\ &+(m-i)\,(2t^i+t^{m+i})\,(t^{m-i}+t^{i-m})+2(m-i).\end{aligned} \tag{15.8}$$

From this exprsssion for a'_i we see (noting that since $t<1$, $i<m$ we have $t^{i-m} > t^{m-i}$) that a' is always positive for $i<m$.

Further writing out at length the expression (15.8) for a'_i we have

$$\begin{aligned}2a^3b^3a'_i \equiv\ &2i\,t^{2i-m} + (m+i)\,t^{2i}\\ &+2(m-i)\,t^{2i-m} + (m-i)\,t^{2i}\\ &-2i\,t^m - (m+i)\,t^{2m}\\ &+2(m-i)\,t^m + (m-i)\,t^{2m} + 2(m-i)\end{aligned}$$

or

$$\begin{aligned}a^3b^3a'_i &= mt^{2i-m}+mt^{2i}+(m-2i)\,t^m-i\,t^{2m}+m-i\\ &= m\,[(1+t^m)+t^{2i-m}\,(1+t^m)]-i\,(1+2t^m+t^{2m})\\ &= (1+t^m)\,[m(1+t^{2i-m})-i\,(1+t^m)].\end{aligned} \tag{15.9}$$

Hence noting that

$$\frac{1}{P}\frac{dP}{d\rho}=\frac{m}{2ab}\cdot\frac{1-t^m}{1+t^m};\quad \frac{P_{2i}}{P^2}=\frac{t^{m-i}\,(1+t^{2i})}{(1+t^m)^2}$$

$$T_{2i}=\tfrac{1}{2}(1-t^2)-\frac{1}{2i}(1+t^{2i})$$

and remembering that the equation of the Elliptic Cylinder at the point of bifurcation is given by

$$T_m \equiv \tfrac{1}{2}(1 - t^2) - \frac{1}{m}(1 + t^m) = 0 \tag{15.10}$$

so that
$$T_{2i} = \frac{1}{m}(1 + t^m) - \frac{1}{2i}(1 + t^{2i})$$

we get for (14.14) the following expression:—

$$\frac{1}{4ia^3b^3}\,\frac{t^{m-i}(1 + t^{2i})}{1 + t^m}\left[m(1 + t^{2i-m}) - i(1 + t^m)\right]$$

$$+ \frac{m}{2a^3b^3}\,\frac{1 - t^m}{1 + t^m}\left[\frac{1}{m}(1 + t^m) - \frac{1}{2i}(1 + t^{2i})\right] \times$$
$$[t^i + \tfrac{1}{2}t^{m+i} + \tfrac{1}{2}t^{m-i}].$$

This can be written as

$$I_1 - I_2 + I_3 - I_4$$

where

$$I_1 = \frac{mt^{m-i}(1 + t^{2i})(1 + t^{2i-m})}{4ia^3b^3\quad\quad 1 + t^m};$$

$$I_2 = \frac{t^{m-i}}{4a^3b^3}(1 + t^{2i});$$

$$I_3 = \frac{1 - t^m}{2a^3b^3}\left[t^i + \tfrac{1}{2}t^{m+i} + \tfrac{1}{2}t^{m-i}\right];$$

$$I_4 = \frac{mt^{m-i}}{4ia^3b^3}\,\frac{1 + t^{2i}}{1 + t^m}(1 - t^m)\left[\frac{1 + t^{2i} + 2t^{2i-m}}{2}\right].$$

It is therefore easy to see that

$$I_1 > I_4 \text{ and } I_3 > I_2$$

whence we get the result that the expression (14.14) and therefore the integral (14.1) is always positive when l is even and equal to $2i$ (the integral (14.1) being zero when l is odd). (15.11)

§ 16. It is now easy to show that ζ_2 can be expressed as a finite expression.

In fact we need only go back to equations (12.4), (12.5), (12.6) for that purpose.

From these equations we have for $l \neq m$

$$(T_l - T_m) \int H\zeta_2 \cos l\psi \, d\psi = \frac{\Delta}{\pi} \int (\zeta_1, \zeta_1) \cos l\psi \, d\psi$$

$$(T'_l - T_m) \int H\zeta_2 \sin l\psi \, d\psi = \frac{\Delta}{\pi} \int (\zeta_1, \zeta_1) \sin l\psi \, d\psi$$

and for $l = m$, we have

$$\int H\zeta_2 \cos m\psi \, d\psi = 0 \text{ from (11.3)}$$

and $$(T'_m - T_m) \int H\zeta_2 \sin m\psi \, d\psi = \frac{\Delta}{\pi} \int (\zeta_1, \zeta_1) \sin m\psi \, d\psi :$$

also from (9.4) equating to zero the co-efficient of α^2 on the left-hand side when we substitute for ζ its expansion

$$\zeta_1 \alpha + \zeta_2 \alpha^2 + \ldots\ldots + \zeta_n \alpha^n + \ldots\ldots$$

we get

$$\frac{1}{\Delta} \int H\zeta_2 \, d\psi + \tfrac{1}{2} \int \frac{\partial}{\partial \rho} \left(\frac{H}{\Delta}\right) \zeta_1^2 \, d\psi = 0.$$

Thus when the equation of the Elliptic Cylinder at the point of bifurcation is given by

$$T_m \equiv \tfrac{1}{2}(1 - t^2) - \frac{1}{m}(1 + t^m) = 0, \qquad (16.1)$$

we get

$$\int H\zeta_2 \, d\psi = -\frac{\Delta}{2} \int \left\{ \frac{1}{\Delta} - \frac{(a^2 + b^2) H}{2\Delta^3} \right\} \frac{\cos^2 m\psi}{H^2} \, d\psi$$

$$= \frac{-\pi m t^m}{2a^2 b^2}.$$

$$\int H\zeta_2 \cos l\psi \, d\psi = \frac{\Delta}{\pi T_l} \int (\zeta_1, \zeta_1) \cos l\psi \, d\psi \text{ for } l \neq m$$

$$\int H\zeta_2 \sin l\psi \, d\psi = \frac{\Delta}{\pi T'_l} \int (\zeta_1, \zeta_1) \sin l\psi \, d\psi = 0$$

$$\int H\zeta_2 \cos m\psi \, d\psi = 0.$$

Hence if we put

$$H\zeta_2 = \sum_{i=0}^{\infty} (A_i \cos i\psi + B_i \sin i\psi)$$

we get

$$2\pi A_0 = -\frac{\pi m t^m}{2a^2 b^2} = -\frac{\pi m}{2a^2 b^2} + \frac{2\pi}{P} \frac{dP}{d\rho} a_0$$

$$\pi A_m = 0 ; \; \pi A_l = 0 \text{ if } l \text{ is odd}$$

and $$\pi A_l = \frac{\Delta}{T_l} \int (\zeta_1, \zeta_1) \cos l\psi \, d\psi \text{ when } l \text{ is even.}$$

$$B_l = 0.$$

Hence using the expression (14.14) and also (16.1), we get

$$H\zeta_2 = \text{constant} + \frac{1}{P}\frac{dP}{d\rho}\sum_{i=0}^{\infty} a_i \cos 2i\psi$$

$$+ \frac{\Delta}{P^3}\sum_{i=1}^{\infty}\frac{P_{2i}}{T_{2i}}\frac{a'_i}{4i}\cos 2i\psi.$$

But we have already shown that

$$a'_i \equiv 0 \text{ for } i \geqslant m.$$

Hence we see that (noting 14.5 and the remarks in § 13) we have the following *finite* expression for the particular integral τ_1

$$\tau_1 = \frac{1}{P}\frac{dP}{d\rho}\frac{\cos^2 m\psi}{[H(\rho\psi)]^2} + \frac{ab}{P^3}\sum_{i=1}^{m-1}{}' \frac{P_{2i}}{T_{2i}}\frac{a'_i}{4i}\frac{\cos 2i\psi}{H(\rho\psi)} \qquad (16.2)$$

wherein the term corresponding to $2i = m$ when m is even is to be omitted, (and the accent denotes this).

§ 17. Instead of defining the new figure (not necessarily one of equilibrium) by means of the equation (5.5), *viz.*

$$U + \frac{\omega^2}{2}(x^2 + y^2) \equiv K\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \text{constant}$$

on its surface we can define it in other ways which may have their own peculiarities. For example we can define it by making the condition that on its surface we must have

$$U + \frac{\omega^2}{2}(x^2 + y^2) \equiv AY + \text{constant} \qquad (17.1)$$

where A is a constant and Y represents that harmonic which has the greatest effect at the point of bifurcation.

Thus if the given cylinder is defined by $T_m = 0$, we take

$$Y = \cos m\psi.$$

Here the condition that the new surface should be one of equilibrium is that

$$A = 0.$$

In fact in whatever way the adjacent surface is defined the condition that it should be one of equilibrium is one and the same, namely, that on its surface

$$U + \frac{\omega^2}{2}(x^2 + y^2) = \text{constant},$$

and therefore whatever method we may follow to define this surface at first, the fact that we shall ultimately want it to be one of equilibrium (and therefore satisfy the above equation), will reduce the conditions in the various cases to the same equation. In the one case we have taken, it reduces to the condition L = 0.

In the other it is A = 0. We shall show later that they are one and the same equation.

As each of the two methods mentioned above of defining the adjacent surface (not necessarily one of equlibrium) leads to some simplification of the final problem we shall develop both and use both methods to get the necessary results for which one method is sometimes more useful than the other and finally when the figure we are in search of is one of equilibrium we shall show that the two conditions are identical.

We have shown how the ζ_n may be calculated without any ambiguity by following the method in which we define the adjacent figure by equation (5.5). We can do a similar thing for the definition given by (17.1) *i.e.*, by means of

$$U + \frac{\omega^2}{2}(x^2 + y^2) = A \cos m\psi + \text{constant}.$$

Expanding U as before we can easily get the following equation

$$\frac{2ab}{(a+b)^2} H\zeta + \frac{1}{2\pi}\int \log R^2 (\rho\rho)\, H'\zeta'\, d\psi'$$

$$= \frac{\Delta}{\pi}\Big[(U_2 + U_3 + \ldots\ldots) - A\cos m\psi\Big] + \text{constant}$$

$$= \frac{\Delta}{\pi}\Big[W - A\cos m\psi\Big] + \text{constant, say.} \qquad (17.2)$$

From this we can easily find a symbolic expression for A when the figure F is not one of equilibrium.

For, multiplying both sides by $\cos m\psi d\psi$ and integrating, we get

$$T_m \int H\zeta \cos m\psi \, d\psi = \frac{\Delta}{\pi} \int (W - A \cos m\psi) \cos m\psi \, d\psi$$

Hence since the given cylinder is defined by $T_m = 0$, we get

$$A = \frac{1}{\pi} \int W \cos m\psi \, d\psi. \tag{17.3}$$

Thus the condition that the adjacent figure should be one of equilibrium is

$$\int W \cos m\psi \, d\psi = 0. \tag{17.4}$$

§ 18. So far we have treated the elliptic cylinder E as fixed. Let us now suppose E to be variable and such that it always differs sufficiently little from the fixed Elliptic Cylinder E_0 belonging to the same series as E. Let us put

$$\frac{\omega^2}{2} = \Omega, \tag{18.1}$$

and take Ω_0 to be the value of Ω for the cylinder E_0 and let

$$\Omega = \Omega_0 + \eta$$

be its value for E.

At the same time let us change our notation and denote by $\rho + \delta\rho$, $q + \delta q$ what we have till now denoted by ρ, q so that the semi-axes of the cylinder E are

$$\sqrt{\rho + \delta\rho + q + \delta q}, \quad \sqrt{\rho + \delta\rho}.$$

As regards E_0 we shall take its semi-axes as $\sqrt{\rho + q}$, $\sqrt{\rho}$. So ρ, q are fixed numbers while $\delta\rho$, δq are known functions of the parameter η and these tend towards zero as η tends to zero.

Now remembering that in the process of defining the adjacent figure F we have Ω the same for it as for the cylinder, we see that the figure of comparison we used formerly is the cylinder E and that the definition of the adjacent figure F is therefore that on its surface we must have

$$U + (\Omega_0 + \eta)(x^2 + y^2)$$
$$\equiv K\left(\frac{x^2}{\rho + \delta\rho + q + \delta q} + \frac{y^2}{\rho + \delta\rho}\right) + \text{constant} \tag{18.2}$$

and in all the equations got from it we need only replace ρ, q by $\rho + \delta\rho$, $q + \delta q$. We note also that the ζ has been defined by taking E as the figure of comparison.

Let us now take E_0 as the figure of comparison, and take the equation of F as

$$x = \sqrt{\rho + q + \zeta_0} \cos \psi \, ; \; y = \sqrt{\rho + \zeta_0} \sin \psi \tag{18.3}$$

where we have put ζ_0 so that it may be clear that it is different from the ζ defined before when E was the figure of comparison.

Exactly as before we can get the equation

$$U + (\Omega_0 + \eta)(x^2 + y^2) = \frac{\Delta}{\pi}\left[-\frac{1}{2\pi}\int \log R^2(\rho\rho)\, H'\zeta'_0\, d\psi' - \frac{2ab}{(a+b)^2} H\zeta_0\right] + W + \text{constant}, \tag{18.4}$$

where

$$W = \eta(\rho + q\cos^2\psi + \zeta_0) + U_2 + U_3 + \ldots\ldots \tag{18.5}$$

On the other hand

$$\frac{x^2}{\rho + \delta\rho + q + \delta q} + \frac{y^2}{\rho + \delta\rho}$$

$$\equiv 1 + \left[\frac{(\zeta_0 - \delta\rho - \delta q)\cos^2\psi}{\rho + \delta\rho + q + \delta q} + \frac{(\zeta_0 - \delta\rho)\sin^2\psi}{\rho + \delta\rho}\right]$$

$$= 1 + \frac{\overline{H}(\zeta_0 - \zeta_{00})}{\overline{\Delta}^2},$$

where $\overline{H}$, $\overline{\Delta}$ are what H, Δ become when we replace ρ, q by $\rho + \delta\rho$, $q + \delta q$ and

$$\zeta_{00} = \delta\rho + \frac{\cos^2\psi}{\overline{H}}(\rho + \delta\rho)\,\delta q \tag{18.6}$$

so that ζ_{00} will be what ζ_0 becomes when the figure F reduces to the elliptic cylinder E. Hence the equation defining the figure F, *viz.*

$$U + (\Omega_0 + \eta)(x^2 + y^2) \equiv K\left(\frac{x^2}{\rho + \delta\rho + q + \delta q} + \frac{y^2}{\rho + \delta\rho}\right) + \text{constant}$$

gives the equation

$$\frac{2ab}{(a+b)^2} H\zeta_0 + \frac{1}{2\pi}\int \log R^2(\rho\rho)\, H'\zeta'_0\, d\psi' = \frac{\Delta}{\pi} W - \overline{L}\,\overline{H}(\zeta_0 - \zeta_{00}) + \text{constant} \tag{18.9}$$

where

$$\overline{L} = \frac{\Delta}{\pi}\cdot\frac{K}{\overline{\Delta}^2} = \frac{\Delta K}{\pi\overline{\Delta}^2} \tag{18.10}$$

and W is given by (18.5) and the bars signify that in the expressions having a bar ρ, q are to be replaced by $\rho+\delta\rho$, $q+\delta q$.

We shall investigate the function ζ_0 as a power series in α, η

$$\zeta_0 = \sum_{r+s>0} \zeta_{rs}\, \alpha^r\, \eta^s \tag{18.11}$$

and for $\overline{L}$ we take the series

$$\overline{L} = L_{00} + \sum_{r+s>0} L_{rs}\, \alpha^r\, \eta^s; \tag{18.12}$$

we shall show that the co-efficients $L_{r-1,s}$ can be calculated at the same time as ζ_{rs}, if we assume that the series (18.11), (18.12) are absolutely and uniformly convergent.

§ 19. In the above change of the figure of comparison (from E to E_0), we have not seen what becomes of the parameter α. When E was the figure of comparison, we defined it by the equation

$$\alpha = \frac{1}{\pi}\int H\,\zeta \cos m\psi\, d\psi;$$

or, in our present notation wherein we must replace ρ, q by $\rho+\delta\rho$, $q+\delta q$, we have

$$\alpha = \frac{1}{\pi}\int \overline{H}\,\overline{\zeta} \cos m\,\psi\, d\,\psi,$$

where the bars indicate that ρ, q have been replaced by

$$\rho+\delta\rho,\ q+\delta q.$$

Again when E is the figure of comparison, the equation

$$\zeta = 0$$

which corresponds to E itself, makes $\alpha = 0$.

Thus in our present notation with E_0 as the figure of comparison, if $\alpha = 0$, the figure F reduces to the ellipse E and ζ_0 becomes

$$\sum_{s=1}^{\infty} \zeta_{0s}\, \eta^s; \tag{19.1}$$

and this is what we called ζ_{00} (equation 18.6) so that

$$\zeta_{00} = \sum_{s=1}^{\infty} \zeta_{0s}\, \eta^s. \tag{19.2}$$

On the other hand, we have similarly

$$L_{00} + \sum_{s>0} L_{0s}\, \eta^s = -\frac{\Delta}{\overline{\Delta}}\, \overline{T}_m, \tag{19.3}$$

where the bars have the same signification as before.

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NOTES AND QUESTIONS.

Notes and Questions.

A Test for the Convergency and Divergency of a Series of Positive Terms.

1. This paper contains an elementary arithmetical test for the convergency or divergency of some kinds of series, namely, series of positive 'non-increasing' terms. From one point of view the condition that the terms are non-increasing can be regarded as no restriction at all; for the behaviour of a series of positive terms is not affected in any way, by re-writing the series in a 'non-increasing' manner. When the terms of a series of positive terms cannot be re-arranged in the above manner, it implies that the set (a_n), has a limit point l $(0 < l \leqslant \infty)$ and such series are evidently divergent. Here a_n denotes the nth term of the series.

Just as Cauchy's condensation test is in a sense an arithmetical representation of a particular case of the theorem:

"*The behaviour of* $\int^{\infty} \phi(t)dt$ *is similar to that of* $\int^{\infty} \phi(f(t))\, f'(t)\, dt$",

namely the case $f(t) = k^t$, so also the test given in this paper is an arithmetical representation of Ermakoff's test. [See Bromwich: *Infinite Series*, § 14].

The range of applicability of the tests given in this paper includes the ranges of Cauchy's condensation test and the logarithmic tests, and extends beyond them. In other words, the tests are applicable to all series, to which Cauchy's condensation test or any of the logarithmic tests is applicable, and further, there are series to which the tests of this paper are applicable, but not the other tests mentioned above.

2. THEOREM I: *If* $\Sigma\, a(r)$ *is a series of non-increasing positive terms and if* $k(r)$ *is defined by the equation*

$$2^r a(2^r) = k(r)\, a(r),$$

then $\Sigma\, a(r)$ *is convergent, if*

$$\overline{\lim}\ k(r) < 1.$$

Proof: Let $\phi(1) = 1$

$$\phi(2) = 2^{\phi(1)} = 2$$

$$\phi(3) = 2^{\phi(2)} = 2^2 = 4$$

$$\phi(4) = 2^{\phi(3)} = 2^4 = 16$$

$$\phi(5) = 2^{\phi(4)} = 2^{16} = 65536$$

and $\phi(n) = 2^{\phi(n-1)}$, for $n \geqslant 2$.

185323

We may without loss of generality take for all r,

$$0 \leqslant k(r) \leqslant \eta < 1.$$

The justifiability of the above supposition will become clear from the proof.

Since $a(r)$ is monotonic and non-increasing

$$\sum_{r=2^n}^{2^{n+1}-1} a(r) \leqslant (2^{n+1}-1-2^n+1)\{a(2^n)\} = 2^n\{a(2^n)\}$$

$$\leqslant \eta\{a(n)\}.$$

Hence we have

$$\sum_{r=\phi(n)}^{\phi(n+1)-1} a(r) \leqslant \eta \sum_{\phi(n-1)}^{\phi(n)-1} a(r)$$

$$\leqslant \eta^2 \sum_{\phi(n-2)}^{\phi(n-1)-1} a(r)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\leqslant \eta^{n-1} a(1).$$

$$\therefore \quad \sum_{r=1}^{\infty} a(r) = \sum_{n=1}^{\infty}\left(\sum_{r=\phi(n)}^{\phi(n+1)-1} a(r)\right)$$

$$\leqslant \sum_{n=0}^{\infty} a(1)\,\eta^n$$

$$= a(1)\frac{1}{1-\eta}.$$

Since $a(r)$ is always positive, this proves the theorem.

3. We may improve the theorem slightly, by replacing the condition

$$\overline{\lim}\ k(r) < 1$$

by the broader condition, that for all large r

$$k(r) < 1 - \frac{2}{r}.$$

Then $$\sum_{r=\phi(n)}^{\phi(n+1)-1} a(r) = O\left(\prod_{r=3}^{n}\left(1-\frac{2}{r}\right)\right)a(1),$$

where the constant implied by "O" is independant of n.

It can easily be shown that

$$\sum_{n=3}^{\infty}\left(\prod_{r=3}^{n}\left(1-\frac{2}{r}\right)\right)$$

is convergent. Hence the result follows.

We can further improve the above result by stating that for all large r, any one of the following holds :

$$k(r) \leqslant 1-\frac{1}{r}-\frac{2}{r\log r}$$

$$k(r) \leqslant 1-\frac{1}{r}-\frac{1}{r\log r}-\frac{2}{r\log r\,.\log\log r}$$

and so on. The proof is easy.

4. THEOREM II: *If* $a(r)$ *and* $k(r)$ *have same meaning as in* Theorem I, $\Sigma\, a(r)$ *is divergent if*

$$k(r) \geqslant 2$$

for all large r.

Proof: Without loss of generality we may take $k(r) \geqslant 2$ for all r. Then

$$\sum_{r=2^n+1}^{2^{n+1}} a(r) \geqslant (2^{n+1}-(2^n+1)+1)\, a(2^{n+1})$$

$$= 2^n . a(2^{n+1})$$

$$= \tfrac{1}{2}\, k(n+1)\, a(n+1)$$

$$\geqslant a(n+\ ),$$

since $k(r) \geqslant 2$.

$$\therefore \sum_{r=\phi(n)+1}^{\phi(n+1)} a(r) \geqslant \sum_{r=\phi(n-1)+1}^{\phi(n)} a(r)$$

....

$$\geqslant a(2)$$

$$\geqslant a(1).$$

Hence, since $a(1) > 0$, the result follows.

5. As before, we may replace the condition that $k(r) \geqslant 2$ by any one of the following conditions:

$$k(r) > 2\left(1 - \frac{1}{r}\right)$$

$$k(r) > 2\left(1 - \frac{1}{r} - \frac{1}{r \log r}\right)$$

$$k(r) > 2\left(1 - \frac{1}{r} - \frac{1}{r \log r} - \frac{1}{r \log r \,.\, \log \log r}\right)$$

and so on.

6. When $1 \leqslant \underline{\lim}\ k(r) \leqslant 2$

the behaviour of the series can be either way.

Thus if we take

$$a(1) = 1$$

$$a(2) = \tfrac{3}{4}$$

$$r\, a(r) = \tfrac{3}{2} a(n), \text{ if } r = 2^n$$

$$a(r) = a(2^n), \text{ if } 2^{n-1} < r < 2^n;$$

we have

$$\sum_{r=2^n+1}^{2^{n+1}} a(r) = 2^n\, a(2^{n+1})$$

$$= \tfrac{1}{2} \,.\, 2^{n+1}\, a(2^{n+1})$$

$$= \tfrac{1}{2} \,.\, \tfrac{3}{2}\, a(n+1)$$

$$= \tfrac{3}{4}\, a(n+1).$$

Hence $$\sum_{r=\phi(n)+1}^{\phi(n+1)} a(r) = \tfrac{3}{4} \sum_{r=\phi(n-1)+1}^{\phi(n)} a(r)$$

....

$$= (\tfrac{3}{4})^{n-1}\, a(2).$$

Thus $$\sum_{r=1}^{\phi(n)} a(r) = 1 + \tfrac{3}{4}\left(\sum_{r=0}^{n-2} (\tfrac{3}{4})^r\right)$$

$$\rightarrow 4 \text{ as } n \rightarrow \infty.$$

Hence the series converges to the sum 4.

If however instead of taking

$$a(r) = a(2^n), \text{ if } 2^{n-1} < r < 2^n,$$

we took
$$a(r) = a(2^{n-1}), \text{ if } 2^{n-1} < r < 2^n,$$

we would have

$$\sum_{r=\phi(n)+1}^{\phi(n+1)} a(r) = (\tfrac{3}{2})^{n-1}\, a(2)$$

and the series is then evidently divergent though $k(r)$ retains its old value.

7. One word may be added about the rapidity with which $\sum_{r=1}^{n} a(r)$ tends to its limit (finite or infinite) when

$$0 < h < k(r) < \mathrm{H} < \infty \quad [\textit{i.e.}, k(r) \asymp 1]$$

In the first numerical illustration given, if

$$\mathrm{R}_m = \sum_{r=m+1}^{\infty} a(r),$$

then
$$\mathrm{R}_m > \mathrm{R}(\phi(n+1)), \text{ where } \phi(n) \leqslant m < \phi(n+1)$$
$$> 4 - 1 - \tfrac{3}{4}\left(\sum_{r=0}^{n-1} (\tfrac{3}{4})^r\right)$$
$$> 4(\tfrac{3}{4})^n.$$

Since $\phi(n)$ increases more rapidly than any logarithmico-exponential function, *and since*

$$4 \,.\, (\tfrac{3}{4})^n$$

is a member of the logarithmico-exponential scale, it is not difficult to prove that R_m tends to zero more slowly than any logarithmico-exponential function which tends to zero.

Similarly in the second illustration, if

$$\mathrm{S}(m) = \sum_{1}^{m} a(r),$$

then $\mathrm{S}(m)$ tends to infinity more slowly than any logarithmico-exponential function which tends to infinity.

T. VIJAYARAGHAVAN.

An Elementary Definite Integral.

The integral considered is :

$$\int_0^\theta \frac{dx}{\alpha \sin^2 x + \beta \cos^2 x} = \frac{\theta}{\sqrt{\alpha\beta}} \equiv \frac{\theta}{\text{geometric mean}}$$

where $$\theta = \frac{m\pi}{2}.$$

1. To deduce Laplace's integrals for $P_n(z)$:
we have from above

$$\frac{\theta}{\sqrt{(\alpha - h)(\beta - h)}} = \int_0^\theta \frac{dx}{(\alpha - h)\sin^2 x + (\beta - h)\cos^2 x}$$

$$= \int_0^\theta \frac{dx}{(\alpha \sin^2 x + \beta \cos^2 x) - h}$$

$$= \sum_0^\infty I_{n+1} h^n \qquad \ldots \text{ I}$$

where $$I_n = \int_0^\theta \frac{dx}{(\alpha \sin^2 x + \beta \cos^2 x)^n}$$

Similarly,

$$\frac{\theta}{\sqrt{(1 - \alpha h)(1 - \beta h)}} = \int_0^\theta \frac{dx}{(1 - \alpha h)\sin^2 x + (1 - \beta h)\cos^2 x}$$

$$= \int_0^\theta \frac{dx}{1 - (\alpha \sin^2 x + \beta \cos^2 x) h}$$

$$= \sum_0^\infty K_n h^n \qquad \ldots \text{ II}$$

where $$K_n = \int_0^\infty (\alpha \sin^2 x + \beta \cos^2 x)^n dx$$

Put $$\alpha = \beta^{-1} = e^{\pm i\phi}.$$

Both the left-hand members in I and II become

$$\frac{\theta}{(1-2h\cos\phi+h^2)^{\frac{1}{2}}}$$

which as we know is equal to

$$\theta\sum_0^{\infty} P_n(\cos\phi)\,h^n.$$

Thus from I and II, comparing the co-efficients of h^n, we get

$$\theta\,.\,P_n(\cos\phi) = \Big[I_{n+1}\Big[_{\alpha=\beta^{-1}=\epsilon^{\pm i\phi}} = \Big[K_n\Big]_{\alpha=\beta^{-1}=\epsilon^{\pm i\phi}}$$

$$= \int_0^{\theta} \frac{dx}{\{\cos\phi \mp i\sin\phi\cos 2x\}^{n+1}}$$

$$= \int_0^{\theta} \{\cos\phi \mp i\sin\phi\cos 2x\}^{n}\,dx$$

where $\theta = \frac{m\pi}{2}$.

Thus we get

$$\left.\begin{aligned} P_n(z) &= \frac{1}{\theta}\int_0^{\theta} \frac{dx}{\{z \pm (z^2-1)^{\frac{1}{2}}\cos 2x\}^{n+1}} \\ &= \frac{1}{\theta}\int_0^{\theta} \frac{dx}{\{z \pm (z^2-1)^{\frac{1}{2}}\cos 2x\}^{-n}} \end{aligned}\right\}\theta = \frac{m\pi}{2}.$$

If we take $m = 2$ and hence $\theta = \pi$ we get the usual Laplace's Integrals

$$\left.\begin{aligned} P_n(z) &= \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\{z + (z^2-1)^{\frac{1}{2}}\cos\phi\}^{n+1}} \\ &= \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\{z + (z^2-1)^{\frac{1}{2}}\cos\phi\}^{-n}} \end{aligned}\right\} \quad \ldots \quad \text{III}$$

putting $2x$ equal to ϕ, no change being introduced as is easily seen.

2. To establish

$$\int_0^\pi \frac{dx}{(a+b\cos x)^{n+1}} = \frac{1}{(a^2-b^2)^{n+\frac{1}{2}}} \int_0^\pi (a+b\cos x)^n \, dx.$$

This could be done by the following steps :

$$\frac{\pi}{\sqrt{\alpha\beta}} = \int_0^\pi \frac{dx}{\alpha \sin^2 x + \beta \cos^2 x}$$

i.e., $$\frac{2\pi}{\sqrt{(\alpha+\beta)^2-(\beta-\alpha)^2}} = \int_0^\pi \frac{2dx}{(\alpha+\beta)+(\beta-\alpha)\cos 2x}$$

$\therefore$ $$\frac{\pi}{\sqrt{(\alpha+\beta)^2-(\beta-\alpha)^2}} = \int_0^\pi \frac{dx}{(\alpha+\beta)+(\beta-\alpha)\cos x}$$

i.e., $$\frac{\pi}{\sqrt{a^2-b^2}} = \int_0^\pi \frac{dx}{a+b\cos x}. \qquad \text{(i)}$$

Next, consider

$$\frac{\pi}{\sqrt{(a-h)^2-b^2}} = \int_0^\pi \frac{dx}{(a+b\cos x)-h}$$

$$= \sum_{n=0}^{\infty} \int_0^\pi \frac{dx}{(a+b\cos x)^{n+1}} \cdot h^n. \qquad \text{(ii)}$$

and similarly,

$$\frac{\pi}{\sqrt{(1-ah')^2-b^2h'^2}} = \int_0^\pi \frac{dx}{1-(a+b\cos x)\,h'}$$

$$= \sum_{n=0}^{\infty} \int_0^\pi (a+b\cos x)^n \cdot h'^n. \qquad \text{(iii)}$$

Thirdly, put $h' = \dfrac{h}{a^2 - b^2}$ and we get

$$\frac{\pi}{\sqrt{(1 - ah')^2 - b^3 h'^2}} = \frac{\pi\,(a^3 - b^2)^{\frac{1}{2}}}{\sqrt{(a - h)^2 - b^3}}.$$

Thus from (ii) and (iii), we have at once

$$(a^2 - b^2)^{\frac{1}{2}} \cdot \sum_{n=0}^{\infty} \int_0^{\pi} \frac{dx}{(a + b\cos x)^{n+1}}\, h^n$$

$$= \sum_{n=0}^{\infty} \int_0^{\pi} (a + b\cos x)^n \cdot \frac{h^n}{(a^2 - b^2)^n} \cdot dx.$$

Comparing co-efficients of h^n, we get

$$\int_0^{\pi} \frac{dx}{(a + b\cos x)^{n+1}} = \frac{1}{(a^2 - b^2)^{n+\frac{1}{2}}} \int_0^{\pi} (a + b\cos x)^n\, dx. \quad \ldots \quad \text{IV}$$

Again putting in (IV) above

$$a = \alpha + \beta \text{ and } b = \beta - \alpha,$$

we get

$$\int_0^{\pi} \frac{dx}{(\alpha \sin^2 x + \beta \cos^2 x)^{n+1}} = \frac{1}{(\alpha\beta)^{n+\frac{1}{2}}} \int_0^{\pi} (\alpha \sin^2 x + \beta \cos^2 x)^n\, dx$$

and notice incidently that by putting $n = 0$ we get our fundamental result

$$\int_0^{\pi} \frac{dx}{\alpha \sin^2 x + \beta \cos^3 x} = \frac{\pi}{\sqrt{\alpha\beta}}.$$

185323

3. It is interesting to notice that

$$\int_0^\theta (\alpha \sin^2 x + \beta \cos^2 x)\, dx = \theta . (A)$$

$$\int_0^\theta (\alpha \sin^2 x + \beta \cos^2 x)^{-1}\, dx = \theta . (G)^{-1}$$

and

$$\int_0^\theta (\alpha \sin^2 x + \beta \cos^2 x)^{-2}\, dx = \theta . \{ (G)\, (H) \}^{-1}$$

where $\theta = \frac{m\pi}{2}$ and (A), (G) and (H) signify the Arithmetic, Geometric and the Harmonic means of (α, β) respectively.

G. S. MAHAJANI.

Solutions.

Question 1302.

(S. RAJANARAYANAN) :—ABC is a triangle. D, E, F are points in the sides such that AD, BE, CF are concurrent. If EF, FD meet BC, CA in G and H respectively, prove that

$$EF . FD . CG . AH = GF . HF . CD . AE.$$

Solution by K. O. Shah, Ganesan, V. A. Mahalingam, S. Audinarayanan, P. Seetarama Rao, M. K. Kewalramani, Ram Behari, and I. Totadri Iyengar.

From the harmonic property of the quadrilateral FDCA, it is clear that

$$\frac{EC}{CH} = \frac{AE}{AH}.$$

Also in the quadrilateral HDGE, CF is a transversal cutting the sides.

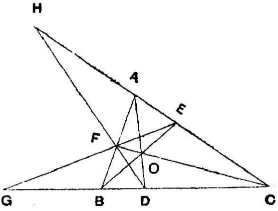
$$\therefore \qquad \frac{HF}{FD} \cdot \frac{DC}{CG} \cdot \frac{GF}{FE} \cdot \frac{EC}{CH} = 1$$

Hence

$$\frac{HF}{FD} \cdot \frac{CD}{CG} \cdot \frac{GF}{EF} \cdot \frac{AE}{AH} = 1$$

i.e.

$$EF . FD . CG . AH = GF . HF . CD . AE.$$



Question 1304.

(N. R. JAYA RAO, M.A.) :—If $\frac{\pi}{2}\,\frac{1.3.5.7\ldots\ldots\ldots(2k-1)}{2.4.6.8\ldots\ldots\ldots 2k} = f(k)$, then show that

$$\int_0^\infty \frac{\sin^{2k+1} x}{x}\,dx = f(k);$$

and

$$\int_0^\infty \frac{\sin^{2k} x}{x}\,dx = f(k-1).$$

Solution by Ram Behari, K. Srinivasa Raghavan, M. K. Kewalramani, N. M. Mehta and K. C. Shah,

(i)

$$\int_0^\infty \frac{\sin^{2k+1}x}{x}\,dx = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^{2k+1}x}{x}\,dx$$

$$= \tfrac{1}{2}\left[\int_0^\pi + \int_\pi^{2\pi} + \int_{2\pi}^{3\pi} + \ldots\ldots\ldots\ldots\right]$$

$$+ \tfrac{1}{2}\left[\int_{-\pi}^{0} + \int_{-2\pi}^{-\pi} + \int_{-3\pi}^{-2\pi} + \ldots\ldots\ldots\ldots\right]$$

the integrand in each case being $\frac{\sin^{2k+1} x}{x}$.

Putting $x = y$, $x = \pi + y$, $x = 2\pi + y$, etc. in the integrals inside the first set of brackets, and $x = y - \pi$, $x = y - 2\pi$, $x = y - 3\pi$, etc. in the integrals inside the second set, we get

$$\int_0^\infty \frac{\sin^{2k+1}x}{x}\,dx = \tfrac{1}{2}\int_0^\pi \sin^{2k+1} y \left\{\begin{array}{l}\frac{1}{y} - \frac{1}{\pi+y} + \frac{1}{2\pi+y} + \ldots\ldots\ldots \\ -\frac{1}{y-\pi} + \frac{1}{y-2\pi} - \ldots\ldots\ldots\end{array}\right\} dy$$

$$= \tfrac{1}{2}\int_0^\pi \sin^{2k+1} y\,.\operatorname{cosec} y\,dy. \quad \text{[Hobson's } \textit{Trig}\text{. Art. 295]}$$

$$= \tfrac{1}{2}\int_0^\pi \sin^{2k} y\,dy = \int_0^{\frac{\pi}{2}} \sin^{2k} y\,dy$$

$$= \frac{\Gamma\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(k+1)}$$

$$= \frac{\pi}{2}\cdot\frac{1.3.5.7\ldots\ldots\ldots\ldots(2k-1)}{2.4.6.8\ldots\ldots\ldots\ldots 2k}.$$

(ii) The second part is incorrect.

Question 1307.

(R. GOPALASWAMI) :—Prove that the Lemoine point and the Euler line are pole and polar with respect to the rectangular hyperbola through the vertices and the centroid.

Solution by S. Audinarayanan, M. K. Kewalramani, N. M. Mehta, and P. R. Venkatakrishna Iyer.

Let K be the Lemoine point, O the orthocentre, and G the centroid of the triangle ABC.

The trilinear co-ordinates of these points are

$$(a, b, c); \left(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right); \text{ and } \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

The equation to a conic through ABC is

$$\frac{F}{x} + \frac{G}{y} + \frac{H}{z} = 0.$$

Since it is a rectangular hyperbola through G, its equation is

$$\begin{vmatrix} \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\ \frac{1}{\cos A} & \frac{1}{\cos B} & \frac{1}{\cos C} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix} = 0.$$

or,
$$\Sigma\, a \cos A\, (b \cos C - c \cos B)\, yz = 0$$

The equation to the polar of (a, b, c) with respect to this is

$$\Sigma\, (c \cos B - b \cos C)\, \frac{\cos A}{a}\, x = 0.$$

This is satisfied by $\left(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right)$ and $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ and therefore represents the Euler line.

Hence the Lemoine point and the Euler line are pole and polar with respect to the rectangular hyperbola through ABCGO.

[NOTE: KG and KO are the tangents from K to this conic.]

Questions for Solution.

1378. (B. B. BAGI) :—Prove that the in-centre of the triangle formed by lines drawn parallel to and at distances $\frac{2}{3}r_1$, $\frac{2}{3}r_2$, $\frac{2}{3}r_3$, from the sides BC, CA, AB of an acute-angled triangle ABC, coincides with the centroid of the ex-central triangle $I_1I_2I_3$; the parallel lines and the vertices being on opposite sides of BC, CA, AB.

1379. (R. GOPALASWAMI) :—(θ)-*pedal triangles and* (θ)-*pedal circles* of a point shall be defined as follows :—Let PD, PE, PF be lines through P parallel to the sides BC, CA, AB of the triangle of reference. If they be rotated about P through an angle θ in the positive direction, so as to meet the sides at D, E, F, then DEF is the (θ)-pedal triangle and the circumcircle of DEF the (θ)-pedal circle of the point P. Shew that

(i) the (θ)-pedal circle of P is the ($-\theta$)-pedal circle of its isogonal conjugate P';

(ii) this circle has double contact with the inconic of the triangle which has P and P' for foci;

(iii) let $D_1E_1F_1$ and $D_2E_2F_2$ be the (θ) and ($-\theta$) pedal triangles of the foci of the inconic of any point O. Let Δ_1 and Δ_2 be triangles whose sides are respectively D_1E_2, E_1F_2, F_1D_2 and D_2E_1, E_2F_1, F_2D_1. Then the triangle of reference is in perspective with each of the triangles Δ_1 and Δ_2 and the centre of perspective is the fixed point O. Examine the cases when θ is a right-angle or O lies on the circumcircle.

1380. (R. GOPALASWAMI) :—Pairs of lines are drawn through the vertices of ABC, parallel to the axes of the parabolas through ABC and any point P, to meet the opposite sides at six points. Shew that the conic through these six points is similar and similarly situated to the inconic of P.

1381. (K. SATYANARAYANA) :—Show that the polars of the conjugates of an involution range with regard to any conic whose directrices pass through the double points form an involution pencil with the focal lines of the vertex of the pencil as double lines.

1382. (K. SATYANARAYANA):—Among the conics which have a given line for directrix and a given point and another given line for pole and polar, show that (1) there is one parabola only and (2) there are two central conics of given eccentricity (3) whose centres are collinear with the given point.

1383. (MARTYN THOMAS):—Two non-intersecting circles are inverted from one of their limiting points into two concentric circles.

If a and b be the radii of the circles, and c the distance between their centres, and k the radius of inversion, prove that the radii of the inverse circles will be

$$\frac{2c^2k^2a}{\Delta(c^2+a^2-b^2+\Delta)} \text{ and } \frac{2c^2k^2b}{\Delta(c^2+b^2-a^2-\Delta)},$$

where $\Delta^2 = a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2$, and find the position of their common centre.

Hence, or otherwise show that, if n circles can be described each touching the two given circles, and also two of their own number,

$$c^2 = a^2+b^2 \pm 2ab\left(\tan^2\frac{\pi}{n}+\sec^2\frac{\pi}{n}\right).$$

1384. (MARTYN THOMAS):—A point P revolves in a circle of radius a with angular velocity ω, while a straight line PQ rotates in the opposite direction with angular velocity ω'. Show that the envelope of PQ is a hypo-cycloid the radii of whose fixed and moving circles are respectively $\left(1+\frac{\omega}{\omega'}\right)a$ and $\frac{\omega'}{2\omega}a$.

Deduce the corresponding result when the straight line PQ rotates in the same direction as P.

1385. (V. TIRUVENKATACHARYA):—Show that, if $q < 1$,

$$(1-q^2)^2(1-q^4)(1-q^6)^2(1-q^8)\ldots\ldots$$
$$\times(1+q^2)(1+q^4)^2(1+q^6)(1+q^8)^2\ldots = 1-q^2-q^6+q^{12}+q^{20}-q^{30}-q^{42}+\ldots\ldots$$

1386. (R. VAIDYANATHASWAMY):—Two triangles $A_1B_1C_1$, $A_2B_2C_2$ are inscribed in the same circle, and their orthocentres are O_1, O_2. Given that A_1' B_1' C_1' A_2' B_2' C_2' are points diametrically opposite to $A_1B_1C_1$ $A_2B_2C_2$; show that, if A_2O_2, B_2O_2, C_2O_2, divide $O_1A'_2$, $O_1B'_2$,

$O_1C'_2$ respectively in the same ratio λ, then A_1O_1, B_1O_1, C_1O_1 also divide O_2A_1', O_2B_1', O_2C_1' respectively in the same ratio μ. Show further that λ and μ are connected by the relation

$$\frac{1}{\lambda} + \frac{1}{\mu} = -2.$$

Prove alse that two triangles related reciprocally as above have the properties :—(*a*) the pedal lines of any point on the circle in regard to them are perpendicular, (*b*) their ortho-centres lie on a diameter. Prove conversely that two inscribed triangles possessing the properties (*a*), (*b*) are necessarily related as above.

1387. (V. RAMASWAMI AIYAR) :—If (x', y') lie on the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

show that the pedal circle of (x', y') with respect to every triangle inscribed in the conic passes through (ξ, η) given by

$$(a + b)x' = (a + b)\xi - 2(a\xi + h\eta + g).$$
$$(a + b)y' = (a + b)\eta - 2(h\xi + b\eta + f).$$

Interpret the result when $a + b = 0$.

1388. (V. RAMASWAMI AIYAR) :—In a parabola, show that the isotomic conjugate of any diameter with respect to any self-conjugate triangle is parallel to the tangent at the extremity ot that diameter.

1389. (K. SATYANARAYANA) :—Denoting

$$c_2c_3 + a(c_2 - c_3) + 3a^2\ ;\ c_3c_1 + a(c_3 - c_1) + 3a^2\ ;\ c_1c_2 + a(c_1 - c_2) + 3a^2$$

by L, M, N, show that the condition that three non-concurrent lines

$$y = m_1 x + c_1\ ;\ y = m_2 x + c_2\ ;\ y = m_3 x + c_3$$

should project into the sides of an equilateral triangle is

$$(M^2 + N^2)(N^2 + L^2)(L^2 + M^2) = 0,$$

the vanishing line being the y axis and the c's being all different.

Interpret geometrically (i) the quantity a, (ii) the absence of m_1, m_2, m_3 in the condition.

195323

LIST OF JOURNALS RECEIVED IN THE LIBRARY.

(during March & April 1925).

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1 Proceedings of the London Math. Soc., Vol. 23, No. 7.
2 Popular Astronomy, Vol. 33, Nos. 2 and 3.
3 Philosophical Magazine, Vol. 49, No. 290, 291.
4 Messenger of Mathematics, Vol. 54, Nos. 1—7.
5 Monthly Notices of the Royal Astronomical Society, Vol. 85, Nos. 1, 2 and 3.
6 Proceedings of the Royal Society, Vol. 107, No. A. 737, 742 and 743.
7 Mathematische Annalen. Band 93, 1/2 and 3/4.
8 Philosophical Transactions, Vol. 224, No. A. 621.
9 Crelle's Journal, Band 154, Heft. 2.
10 Rendiconti del Circolo Matematica di Palermo, Tome 48, Fascicolo II.
11 Annals of Mathematics, Vol. 26, Nos. 1 and 2.
12 Japanese Journal of Mathematics (Transactions and Abstracts), Vol. 1, Nos. 1 and 2.
13 Nature, Vol. 114, No. 2878.
" " 115, Nos. 2879 to 2884; 2885 to 2888 and Index to Volume 114.
14 Liouville's Journal, Volumes of years 1836 to 1907, 1909, 1914 and 1917 to 1924.
15 Mathematische Annalen, Volumes 1 to 62, 72—81 and 86.
16 Bulletin des Sciences Mathematiques { Tome 48, Dec. 1924. Tome 49, Jan. 1925.
17 Annals de l'Ecole Normale Superieure, Vol. 58, Nos. 11 & 12.
18 The American Mathematical Monthly, Vol. 32, Nos. 1 and 2.
19 Encyklopedie des Mathematischen Wissenschaften, Band VI i-B, Heft 5.
20 Transactions of the American Mathematical Society, Vol. 27, No. 1.
21 Bulletin of the American Mathematical Society, Vol. 31, Nos. 1 and 2.

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